MATH 105 101 Midterm 1 Sample 2

1. (15 marks) Let $z = f(x, y) = x^2y^2$.

   (a) (5 marks) Compute all second-order partial derivatives of f.

   Solution: Calculate the first-order partial derivatives of f:

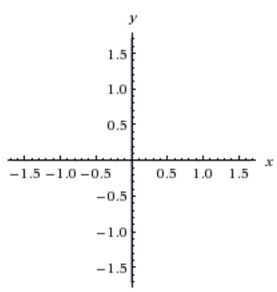
   $$f_x(x, y) = 2xy^2, \qquad f_y(x, y) = 2x^2y.$$

   Compute the four second-order partial derivatives of f:

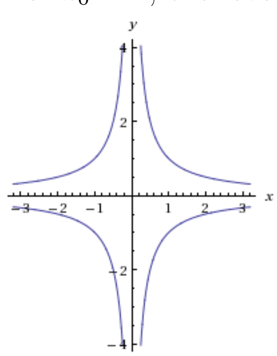
   $$f_{xx}(x, y) = 2y^2, \qquad f_{yy}(x, y) = 2x^2, \qquad f_{xy}(x, y) = f_{yx}(x, y) = 4xy.$$

   (b) (4 marks) Sketch the level curves $f(x, y) = z_0$ with $z_0 = 0$ and $z_0 = 1$.

   Solution: First, note that the domain of f is $\mathbb{R}^2$.
   For $z_0 = 0$, the equations $x^2y^2 = 0$ holds for $x = 0$ or $y = 0$. So, the level curve is the x- and y- axes.

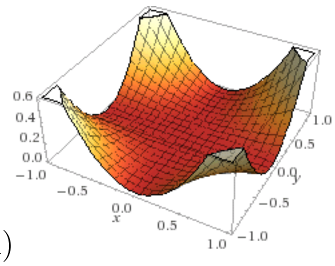


   For $z_0 = 1$, the level curve $x^2y^2 = 1$ consists of 4 curves.

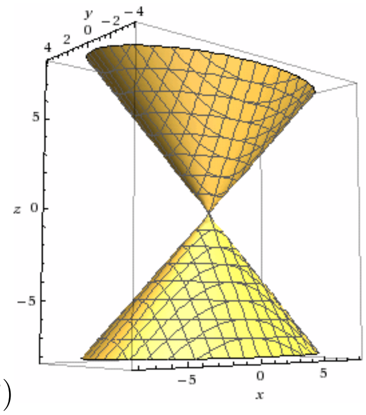


   (c)

(d) (1 mark) Which of the following renderings represents the graph of the surface?



(A)



(B)

Solution: The answer is (A), since $f(x, y) = x^2y^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, and in (B), the graph includes points with negative z-values.

(e) (3 marks) Find an equation of the plane $\mathcal{P}$ passing through the point $(2, -3, f(2, -3))$ with a normal vector $\mathbf{n} = \langle -1, 3, 2 \rangle$. *Simplify your answer.*

Solution: We have $f(2, -3) = 2^2(-3)^2 = 36$. So, an equation of the plane passing through $(2, -3, 36)$ with a normal vector $\mathbf{n} = \langle -1, 3, 2 \rangle$ is:

$$-1(x - 2) + 3(y + 3) + 2(z - 36) = 0$$
$$\Rightarrow -x + 3y + 2z = 61.$$

(f) (2 marks) Does the equation $2x - 6y - 4z = -122$ describe the same plane in (d)? Justify your answer.

Solution: The equation $2x - 6y - 4z = -122$ describes the same plane in (d) because if we multiply the whole equation of the plane in (d) by -2, we obtain exactly the equation $2x - 6y - 4z = -122$. So, these two planes consist of the same set of points, and therefore, must be equal.

2. (5 marks)

(a) (2 marks) Let $f(x, y) = \ln(9 - x^2 - y^2)$. Sketch the domain of f in the xy-plane.

Solution: The only restriction is that $9 - x^2 - y^2 > 0$ so that we can take ln. So, the domain of f is:

$$D = \{(x, y) \in \mathbb{R}^2 \mid 9 - x^2 - y^2 > 0\}.$$

To sketch the domain of f, note that it consists of all points (x, y) lying strictly inside the circle centered at $(0, 0)$ with radius 3. The circle itself should be dotted, as it is not included in the domain.

(b) (3 marks) Show or disprove that there exists a function g which has continuous partial derivatives of all orders such that:

$$g_x = 9998x^{9998}y \text{ and } g_y = x^{9999}.$$

Solution: Suppose that such a function $g(x, y)$ exists. Then,

$$g_{xy} = 9998x^{9998} \neq 9999x^{9998} = f_{yx}.$$

Note that g_{xy} and g_{yx} are both continuous on $\mathbb{R}^2$, being polynomials.Then, $g_{yx} \neq g_{xy}$ contradicts Clairaut's Theorem which states that if g_{yx} and g_{xy} are continuous, then $g_{xy} = g_{yx}$. Therefore, there does not exist a function $g(x, y)$ with the given partial derivatives.

3. (10 marks) Let R be the semicircular region $\{x^2 + y^2 \leq 9, y \geq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 4y$$

on the *boundary of the region R*.

Solution: The boundary of the region R consists of two pieces: the semicircular arc which can be parametrized by $x = 3\cos\theta$ and $y = 3\sin\theta$ for $0 \leq \theta \leq \pi$, and the horizontal segment $y = 0$ for $-3 \leq y \leq 3$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the semicircular arc*: We have that $f(x, y) = g(\theta) = (3\cos\theta)^2 + (3\sin\theta)^2 - 4(3\sin\theta) = 3 - 12\sin\theta$ for $0 \leq \theta \leq \pi$. Then, $g'(\theta) = -12\cos\theta = 0$ if and only if $\theta = \frac{\pi}{2}$, which correspond to $x = 0$ and $y = 3$. So, there are 3 points where extrema can occur: $(0, 3)$ (critical point), $(3, 0)$ and $(-3, 0)$ (end points).

- *On the horizontal segment*: We have that $f(x, 0) = h(x) = x^2$ for $-3 \leq x \leq 3$. So, $h'(x) = 2x = 0$ when $x = 0$. So, there are 3 points where extrema can occur: $(0, 0)$ (critical point), $(3, 0)$ and $(-3, 0)$ (end points).

Evaluate f at those points, we get:

$$f(0, 3) = -3, \quad f(3, 0) = f(-3, 0) = 9, \quad f(0, 0) = 0$$

Thus, on the boundary of R, f attains the absolute maximum value 9 at the points $(3, 0)$ and $(-3, 0)$ and the absolute minimum value -3 at the point$(0, 3)$.

4. (10 marks) Find *all* critical points of the following function:

$$f(x, y) = 3x^2 - 6xy + y^3 - 9y$$

Classify each point as a local minimum, local maximum, or saddle point.

Solution: Compute the first-order partial derivatives of f:

$$f_x(x, y) = 6x - 6y = 6(x - y) \qquad f_y(x, y) \qquad = -6x + 3y^2 - 9$$

Since both f_x and f_y are defined at every point in $\mathbb{R}^2$, the only critical points of f are those at which $f_x = f_y = 0$. If $f_x = 0$, then $x = y$. Put $x = y$ into $f_y = 0$, we get $-6y + 3y^2 - 9 = 3(y - 3)(y + 1) = 0$, which gives $y = 3$ or $y = -1$. So we get two critical points $(3, 3)$ and $(-1, -1)$.

Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = 6, \quad f_{yy} = 6y, \quad f_{xy} = -6, \quad D(x, y) = 36y - 36$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(3, 3)$, $D(3, 3) = 72 > 0$ and $f_{xx}(3, 3) = 6$, so $(3, 3)$ is a local minimum.
- At the point $(-1, -1)$, $D(-1, -1) = -72 < 0$, so $(-1, -1)$ is a saddle point.

5. (10 marks) A company wishes to build a new warehouse. It should be situated on the northeast quarter of the Oval, an expressway whose shape is given by the equation:

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

Here x and y are measured in kilometers. From the company's point of view, the desirability of a location on the Oval is measured by the sum of its horizontal and vertical distances from the origin. The larger the sum is, the more desirable the location is. Using the method of Lagrange multipliers, find the location on the Oval that is the most desirable to the company. Clearly state the objective function and the constraint. *You are not required to justify that the solution you obtained is the absolute maximum.* **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

Solution: Since the desirability of a location on the Oval is measured by the sum of its horizontal and vertical distances from the origin, we get the objective function $f(x, y) = x+y$ which we want to maximize. The location should be on the northeast quarter of the oval, so the constraint function is $g(x, y) = \frac{x^2}{9} + \frac{y^2}{16} - 1 = 0$ and we want $x \geq 0, y \geq 0$. Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned} \nabla f(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= 0 \end{aligned}$$

More explicitly, we need to solve:

$$\begin{aligned} 1 &= \frac{2x\lambda}{9} \\ 1 &= \frac{2y\lambda}{16} \\ \frac{x^2}{9} + \frac{y^2}{16} - 1 &= 0 \end{aligned}$$

Equating the first two equations, we get:

$$\frac{2x\lambda}{9} = 1 = \frac{2y\lambda}{16} \Rightarrow y = \frac{16}{9}x.$$

Substitute that into the last equation, we get:

$$\frac{x^2}{9} + \frac{y^2}{16} - 1 = 0 \Rightarrow \frac{x^2}{9} + \frac{16x^2}{81} - 1 \qquad = 0 \Rightarrow \frac{25x^2}{81} = 1 \Rightarrow x = \pm\frac{9}{5}.$$

Since we want only $x \geq 0$, we get $x = \frac{9}{5}$. Then, $y = \frac{16}{5}$ and $\lambda = \frac{5}{2}$. Thus, the most desirable location is $\left(\frac{9}{5}, \frac{16}{5}\right)$.